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INTERACTION OF TWO-DIMENSIONAL TRAILING VORTEX PAIR WITH A SHEAR LAYER*

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The basic laws governing the interaction of a two-dimensional vortex pair with a shear layer of constant thickness are considered. The main idea of the study is to develop and adapt a simplified representation of a hydrodynamic flow based on a point-vortex model simulating the actual interaction of full-scale vortex patterns over the ground surface. It is shown that vortices with vorticity opposite in sign to the shear layer may stop or even ricochet from this layer, while the other vortex may penetrate through the layer. Numerical results are presented as plots and analyzed.

Introduction. Two clearly defined approaches to the solution and analysis of applied problems have recently evolved in modern hydrodynamics. The first approach involves a strict solution of the problem of interest. In this area of hydrodynamics, an applied problem is formulated in detail, various physical fields and processes and their interaction are allowed for, the basic equations relating physical fields and the parameters of the problem are substantiated and derived, and modern methods for solution and analysis of problems of mathematical physics are applied. The second approach involves a development of various simplified model approximations describing the dynamic interaction of actual hydrodynamic flows.

While the former area calls for significant efforts to solve the problem posed, considerable computational resources, and refined and adaptive computational methods, the latter approach does not set such requirements and the problem can be solved in real-time mode on a modern computer (for example, within the framework of the ideal incompressible fluid model).

These approaches are interrelated. By solving exactly a physical problem, we can reveal the basic laws of hydrodynamic processes, the influences of various effects and their development in time, the range of the fundamental effect of one physical phenomenon or another, and the effect on the flow evolution. However, in most cases, such problems turn out to be labor-consuming and run into (sometimes insuperable) difficulties; therefore, some problems have not been solved or analyzed yet. At the same time, the second approach based on already solved problems of applied hydromechanics allows us to give fairly quickly a qualitative (faithful from the practical viewpoint) and, in some cases, quantitative description of complex hydrodynamic processes and to analyze the domain of applicability of one model approach or approximation or another. Such an analysis turns out to be quite useful and efficient for development of fundamental theories and approaches to the solution of problems of mathematical physics, which constitute the basis of the first scientific area.

It is known [5, 6, 11] that a take-off of a large-capacity aircraft is accompanied by complex air motion. It is important that the aircraft wings are followed by a pair of large-scale trailing vortices, which stay over the runway quite a long time, even under a near-surface crosswind. The intensity of such vortices is rather high. For example, after a Boeing 747 has taken off, the vortices generated have intensity of approximately 250 m^2/sec and are spaced approximately 25 m apart [6]. Such vortices may hang over the runway for several minutes, even with a crosswind speed of 3–5 m/sec. This complex air motion is rather dangerous to the subsequent aircraft either taking off or landing. In this connection, the practically important question

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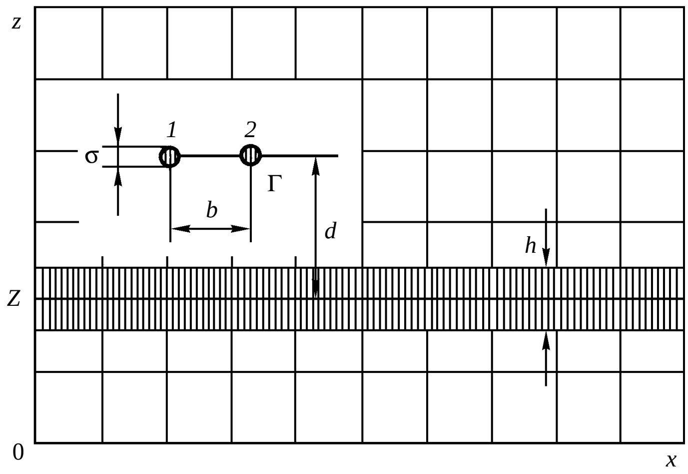

Fig. 1

arises: What time does it take the pair of large-scale vortices to leave the take-off area in the presence of an average near-surface crosswind of known profile?

The present paper treats the problem on a two-dimensional vortex pair interacting with a shear layer of constant thickness with homogeneous vorticity distribution. The problem is solved exactly by solving the two-dimensional Navier–Stokes equation with a prescribed initial velocity field generated based on actual data [6]. The subsequent flow analysis is based on a quite simple model describing the interaction of two point vortices over a shear layer formed by discrete point vortices and a distributed vorticity field of constant thickness. Both solutions are compared and analyzed to create a simple and yet quite truthful flow model, which can be applied in real-time mode.

1. Formulation of the Problem. Let us consider a viscous two-dimensional incompressible fluid of density ρ and viscosity ν moving over a hard surface $z = 0$. Let a pair of vortices (Fig. 1) with local vorticity and centers with coordinates (x_1, z_1) and (x_2, z_2) be at some height at the initial instant of time $t = 0$. It is assumed that the distribution of the velocity field $U(x, z, 0)$ and, hence, the vorticity field $\omega(x, z, 0)$ is known at this instant. It is necessary to determine the evolution of the vortex pair in time.

The motion of a viscous homogeneous incompressible fluid is described by the vector equation expressing the law of preservation of momentum and the equation of continuity

$$\frac{\partial \vec{U}}{\partial t} + (U \cdot \nabla) \cdot \vec{U} = \frac{1}{\rho} \nabla p + \nu \Delta \vec{U}, \tag{1}$$

$$\nabla \cdot \vec{U} = 0. \tag{2}$$

Here ∇ is Hamiltonian, Δ is Laplacian, p is pressure, ρ is the fluid density, and $\vec{U}\left[U_x(x, z, t), U_z(x, z, t)\right]$ is the velocity field.

The problem formulation should be supplemented with initial conditions: the distribution of the velocity field in the flow

$$\vec{U}(x, z, 0) = \vec{U}^{(0)}(x, z), \tag{3}$$

including the distribution of the velocity field in each vortex constituting the pair

$$\vec{u}_\omega(x, z, 0) = \vec{U}_\omega^{(0)}(x, z). \tag{4}$$

We assume that at the initial instant the distribution of the vortex-pair velocity field does not affect the distribution of the external-flow velocity field. This statement is true if the vortex pair is initially far from the region of a flow with high velocity gradients, such as the region over a shear layer where the velocity field tends to be constant.

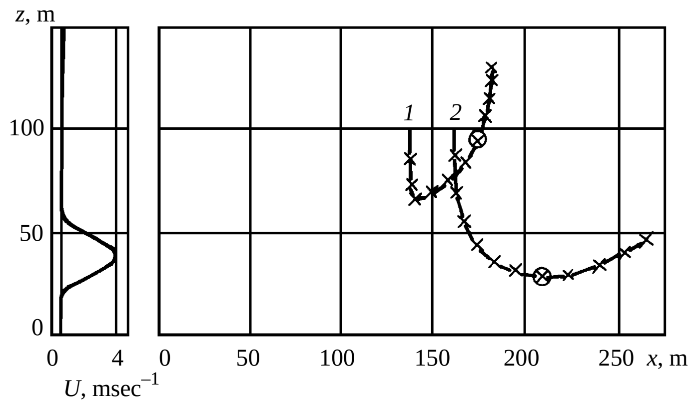

Fig. 2

It is also necessary to define boundary conditions on the surface and at infinity, which may be written as

$$\vec{u}(x, 0, t) = 0, \quad \vec{U}(x, z, t) = 0 \quad \text{as} \quad (x \to \infty,\ z \to \infty). \tag{5}$$

Experimental data show [6, 7] that after a take-off of a heavy aircraft, the vorticity distribution in the vortex pair agrees well with the distribution of the vorticity field in Lamb–Oseen vortices.

In this case, the velocity field $\vec{U}_\omega$ has only the circumferential component, which can be represented as [8]

$$V_\theta(r) = \frac{V^{(0)}}{2\pi r}\left(1 - \exp\left[-\beta\left(\frac{r}{r_c}\right)^2\right]\right), \tag{6}$$

where r is the distance from the vortex center to the field point of interest, r_c is the characteristic dimension of a vortex, and $V^{(0)}$ is the characteristic velocity.

The initial distribution of the velocity field of the near-surface airflow (Fig. 1) can be represented in the form

$$U_z(x, z, 0) = 0, \quad U_x(x, z, 0) = \begin{cases} 0, & 0 < z \le Z - h/2 \\ U_x^{(0)}(z), & Z - h/2 < z \le Z + h/2, \\ U^{(0)}, & z > Z + h/2 \end{cases} \tag{7}$$

which is a shear flow in a layer of height h with the center located at height Z. Here $U^{(0)}$ is the maximum flow velocity over the shear layer and $U_x^{(0)}(z)$ is a function describing the initial velocity distribution in the layer. Certainly, the velocity field is a continuous function: $U_x^{(0)}(Z - h/2) = 0$, $U_x^{(0)}(Z + h/2) = U^{(0)}$.

2. Numerical Analysis. Numerical results are obtained in solving the Navier–Stokes (1) and continuity (2) equations together with the initial and boundary conditions (3)–(5). The inertial and viscous terms of the equations were integrated using the Adams–Bashforth process of the second order [3, 7] with an additional diagnostic step at which Poisson's equation was solved to determine the distribution of the dynamic-pressure field [10]. Partial derivatives were determined using central differences of the second order on a mesh covering the field under study. The boundary conditions along the flow and in the opposite direction are selected periodic. On the other hand, the boundary conditions along the axis OZ are determined by the vertical scale of the problem.

Figure 2 depicts the results of numerical simulation of two interacting identical vortices spaced at $b = 25$ m at the initial instant and having intensity $\Gamma = 250$ m^2/sec and vorticity concentrated in a circular domain of radius $\sigma = 1.66$ m for $\beta = 1.1264$ (formula (6)). Initially, the shear layer is a horizontal airflow located at height $Z = 40$ m and having total height $h = 60$ m and the maximum velocity $\max\left\{U_x^{(0)}\right\} = 4$ m in the middle part. The positions of the vortices at regular intervals $\Delta t = 10$ sec are represented in Fig. 2 by crosses.

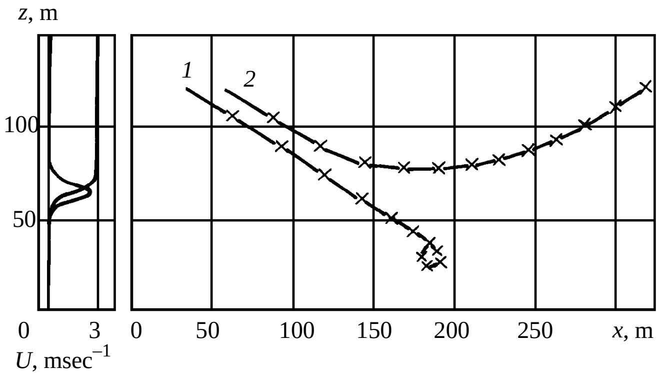

Fig. 3

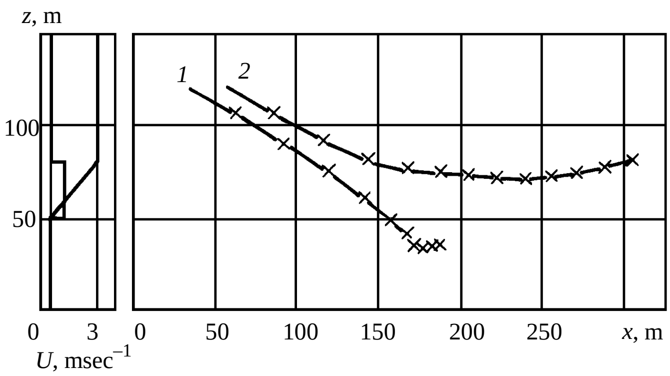

Fig. 4

The vortex pair moves toward the ground surface. Eventually, vortex *1* with intensity of sign coinciding with that of the vorticity in the shear layer begins to interact and stops at $t \approx 30$ sec. On the other hand, vortex *2* penetrates through the layer. After that, it begins to interact with the surface and, therefore, moves together with the flow, remaining under the layer all the time. At the same time, vortex *1* is pushed by the layer vertically upwards — beyond the zone of interaction. The position of the vortices at $t = 70$ sec is shown in Fig. 2 by circles. The numerical example shows that, in fact, the vortex pair leaves the zone of interaction in $t \approx 100$–120 sec, vortex *2* passing a distance of approximately 100 m and vortex *1* rising at a height of 125 m.

A similar case of interaction is depicted in Fig. 3. Here, the initial velocity profile of the shear layer is slightly changed, the other parameters remaining the same:

$$\Gamma = 250\ \mathrm{m}^2/\mathrm{sec}, \quad b = 25\ \mathrm{m}, \quad \sigma = 16.6\ \mathrm{m}, \quad \beta = 1.1264, \quad L = 65\ \mathrm{m}, \quad h = 20\ \mathrm{m}, \quad \max\{U_x^{(0)}\} = 3\ \mathrm{m}.$$

Note that the interaction pattern has not changed qualitatively. A global drift of the vortices with the flow is observed. Vortex *2* with vorticity identical in sigh with that of the shear layer remains over the layer and leaves the zone of interaction with a rather high velocity due to the formation of vortex pair with the shear layer. On penetrating the layer, vortex *1* remains under the layer nearly motionless.

It is interesting that if the velocity profile of the shear layer is linearized, the qualitative pattern of interaction (especially at the initial stage) of the vortex pair with the shear layer does not change (Fig. 4). In this case, the initial data correspond to the previous case (Fig. 3). This example allows us to conclude that when the vortex pair interacts with the layer, the form of the velocity profile does not affect significantly its evolution. The numerical analysis shows that the following quantitative characteristics of interaction are important in this case: the intensity of the vortex pair, its initial position, and the value of the jump in the velocity of the shear layer.

3. Model Approximation. A detailed analysis of the numerical solution of the problem being considered indicates that viscous effects have no significant value at the initial stage of vortex interaction, since the vertical and horizontal velocity gradients are small during the interaction: the inertia effects dominates over the dissipative effects, and the viscous terms in Eqs. (1) can be neglected on the time interval involved. Hence, in creating an interaction model, it is possible to make use of the ideal incompressible fluid model.

The results obtained allow us to conclude that the region of the shear flow nearest to the vortex pair influences most the layer–pair interaction. The distant regions almost do not participate in the interaction. However, the distant flow region should not be neglected. It is known [11] that, in this case, the shear flow swirls, which finally leads to deformation of the basic (nearby) region of the layer and, consequently, to a quantitative and qualitative change in the evolution of the vortex pair.

It is convenient to represent the layer by a system of moving point vortices (N layers, each containing M vortices) with intensity $\Gamma^{(0)} = \mathrm{const}$ and by fixed vortex bands with homogeneous distributed vorticity $\omega^{(0)} = \mathrm{const}$ (Fig. 5). Then, according to the Biot–Savart law, the vertical component of the velocity at an arbitrary point of the field (x, y) has the form [8]

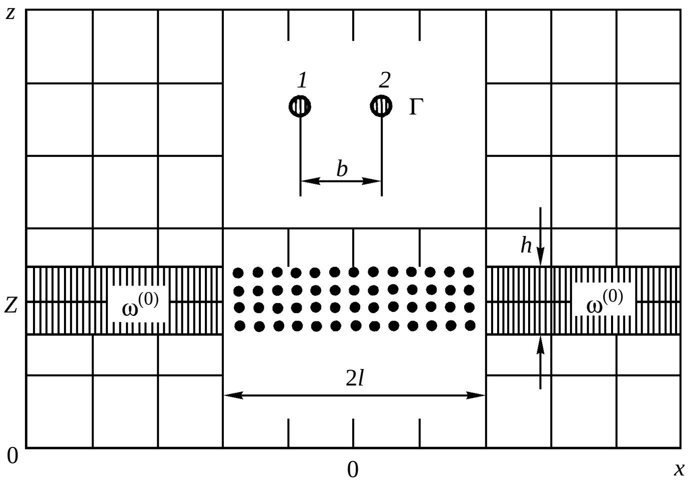

Fig. 5

$$U_z(x,z) = \frac{1}{2\pi}\int\limits_{l}^{\infty}\int\limits_{-h/2}^{h/2}\omega(z')\frac{x-x'}{(x-x')^2+(z-z')^2}\,dz'\,dx'$$

$$= \frac{\omega^{(0)}}{4\pi}\left\{F_z\left(\frac{h}{2}, x, z\right) - F_z\left(-\frac{h}{2}, x, z\right)\right\}, \tag{8}$$

where

$$F_x(z', x, z) = (z'-z)\ln\frac{(z'-z)^2+(l-x)^2}{(z'-z)^2+(l+x)^2}$$

$$+ 2(l+x)\arctan\frac{z'-z}{l+x} - 2(l-x)\arctan\frac{z'-z}{l-x}. \tag{9}$$

The horizontal component of the velocity can be determined similarly:

$$U_x(x,z) = -\frac{1}{2\pi}\int\limits_{l}^{\infty}\int\limits_{-h/2}^{h/2}\omega(z')\frac{z-z'}{(x-x')^2+(z-z')^2}\,dz'\,dx'$$

$$+ \frac{1}{2\pi}\int\limits_{-\infty}^{-l}\int\limits_{-h/2}^{h/2}\omega(z')\frac{z-z'}{(x-x')^2+(z-z')^2}\,dz'\,dx'$$

$$= F(z) - \frac{\omega^{(0)}}{2\pi}\left\{F_x\left(\frac{h}{2}, x, z\right) - F_x\left(-\frac{h}{2}, x, z\right)\right\}, \tag{10}$$

where

$$F(z) = \begin{cases} -\dfrac{\omega^{(0)}h}{2}, & z > Z + h/2 \\ -\omega^{(0)}(z - Z), & Z - h/2 < z < Z + h/2 \\ \dfrac{\omega^{(0)}h}{2}, & z < Z - h/2 \end{cases},$$

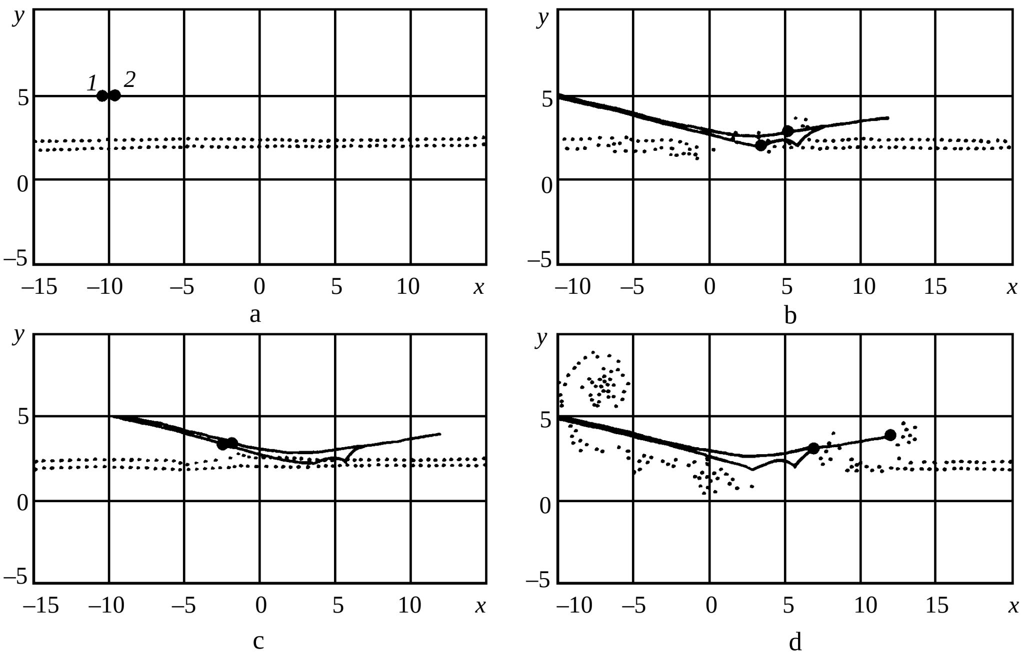

Fig. 6

$$F_x(z', x, z) = (z' - z)\arctan\frac{l - x}{z' - z} + \frac{l - x}{2}\ln\left[1 + \left(\frac{z' - z}{l - x}\right)^2\right]$$

$$+ (z' - z)\arctan\frac{l + x}{z' - z} + \frac{l + x}{2}\ln\left[1 + \left(\frac{z' - z}{l + x}\right)^2\right]. \qquad (11)$$

The velocity field at a point (x', z'), which is induced by every point vortex with intensity $\Gamma^{(0)}$ at the current point (x, z), is determined from the expressions [4, 9]

$$U_x(x', z') = \frac{1}{4\pi}\Gamma^{(0)}\frac{z - z'}{(z' - z)^2 + (x - x')^2},$$

$$U_z(x', z') = -\frac{1}{4\pi}\Gamma^{(0)}\frac{x - x'}{(z' - z)^2 + (x - x')^2}. \qquad (12)$$

The effect of the plane boundary is simulated by symmetrically mapping the flow about the plane $z = 0$. Thus, once the initial position of the vortex pair ($Z^{(0)}$, $X^{(0)}$, b), its intensity Γ, and the initial parameters of the vortex layer (l, $N \times M$, h, Z, and $\omega^{(0)}$) are specified, the problem is defined completely and can be analyzed numerically using expressions (8–12).

4. Analysis of the Model Approximation. Let us consider how a vortex pair interacts with a shear flow whose initial parameters coincide with those in Fig. 4. Let all the geometrical parameters of the flow be referred to the height h of the shear layer and the vorticity field to the maximum (in absolute value) vorticity in the shear layer $|\omega^{(0)}|$. The velocity of the vortex pair can be normalized to $h\,|\omega^{(0)}|$, time to $|\omega^{(0)}|^{-1}$, and the vortex intensity to $|\omega^{(0)}|\,h^2/2\pi$.

Figure 6 shows how the vortex pair interacts with the shear flow, whose vorticity is simulated by a system of equidistant vortices ($N \times M = 2 \times 120$) located in a rather wide region ($l = 30.0$) of the layer (outside this region, a region of fixed vorticity $\omega^{(0)} = -1.0$ is located). The initial position of the system is shown in Fig. 6a. Eventually, the vortices come nearer to the shear layer, keeping the spatial orientation and the relative distance between the vortices. The position of the vortex pair at the time $t = 8.0$ is shown in Fig. 6b by filled circles. After that, vortex *1* with positive vorticity entraps some nearby vortices of the shear layer, forming a new vortex pair (see Fig. 6c corresponding to dimensionless $t = 16.0$) and moves

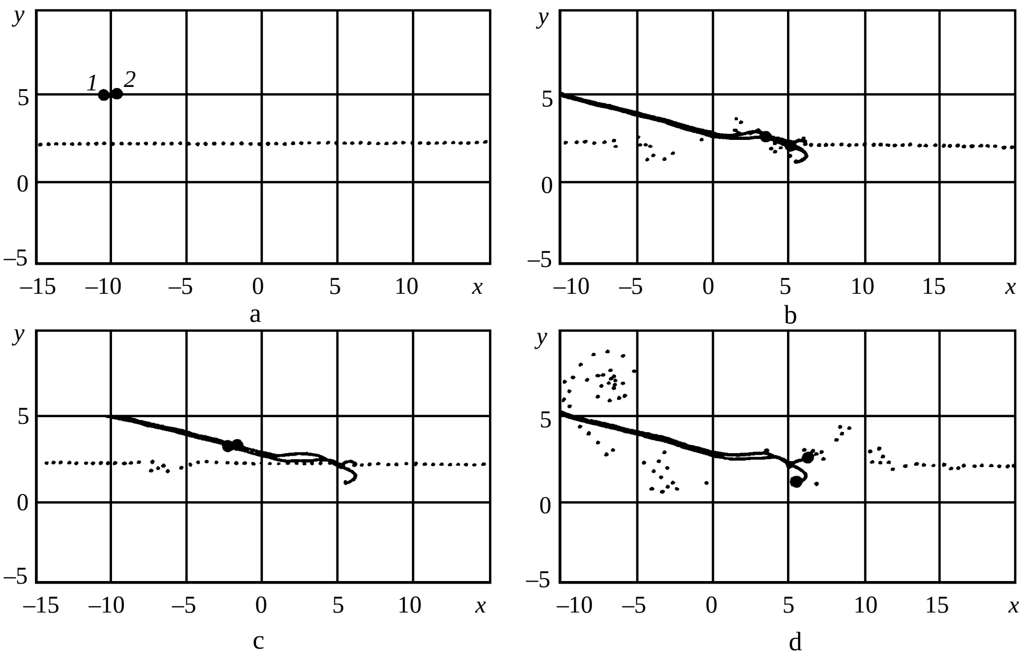

Fig. 7

beyond the region in question, remaining over the layer all the time (Fig. 6d, t = 24.0). On the other hand, vortex *1* with negative vorticity interacts with the shear layer, its drift velocity being considerably lower than that of vortex *2*.

A decrease in the number of layers of point vortices in the shear layer changes significantly the trajectories of the vortices under consideration. This case of interaction is depicted in Fig. 7. The indices in these (as well as subsequent) figures correspond to the instants of time specified in Fig. 6. A decrease in the number of layers leads to an increase in the vorticity of the point vortices constituting the layer. They begin to interact with the initial vortex pair more intensively, which most likely causes the vortex pair to hang over the central region of the shear flow.

In contrast to this case, an increase in the number of layers of point vortices forming the shear flow leads to a decrease in the intensity of point vortices and, hence, to a more accurate description of the height vorticity profile (Fig. 8). As a result, the evolution of the vortex pair agrees better with the numerical solution of the viscosity-corrected equations of motion (Fig. 5).

Figure 9 illustrates the interaction of the vortex pair with the system of point vortices with constant density of filling the middle part of the flow ($N \times M = 4 \times 40$) by point vortices and with smaller $l = 20.0$ compared with the previous cases. A numerical experiment shows that vortex *1*, interacting, remains over the layer all the time, while vortex *2* of the initial vortex pair tends to penetrate through the layer, displacing in the positive direction of the OX axis. A qualitative comparison of the results allows us to conclude that the width of the active part of the layer filled with point vortices does not affect significantly the evolution of the initial vortex pair.

The evolution of the vortex system is affected greater by the density of filling the layer by point vortices. The case $N \times M = 4 \times 30$ (for $l = 30.0$) is shown in Fig. 10. A simple comparative analysis shows that the qualitative interaction pattern of the vortex pair changes at the final stage — vortex *1* penetrates under the layer.

Conclusions. The studies made allow us to conclude that the ideal incompressible fluid model can be used in a qualitative analysis of the interaction of a vortex pair with a shear layer of surface air. Changes in the evolution of the vortex pair, especially at the final stage of interaction, are attributed to viscosity. The viscous effects are insignificant at the initial stage of interaction, which includes the approach of the vortex pair to the shear layer, penetration of one vortex under the layer, and ricochet of the other vortex from the layer. This is accounted for by the fact that the distribution of the velocity field in the region nearest to the layer has small gradients of the velocity profile. As a result, the inertia effects dominate over the viscous effects quite a long time, which is estimated in [1] for a system of two-dimensional vortex patterns.

Numerical experiments show that the model proposed for studying the evolution of a vortex pair over a layer can be used as a preliminary, approximate analysis of the problem. However, the number of parameters of this model is greater than

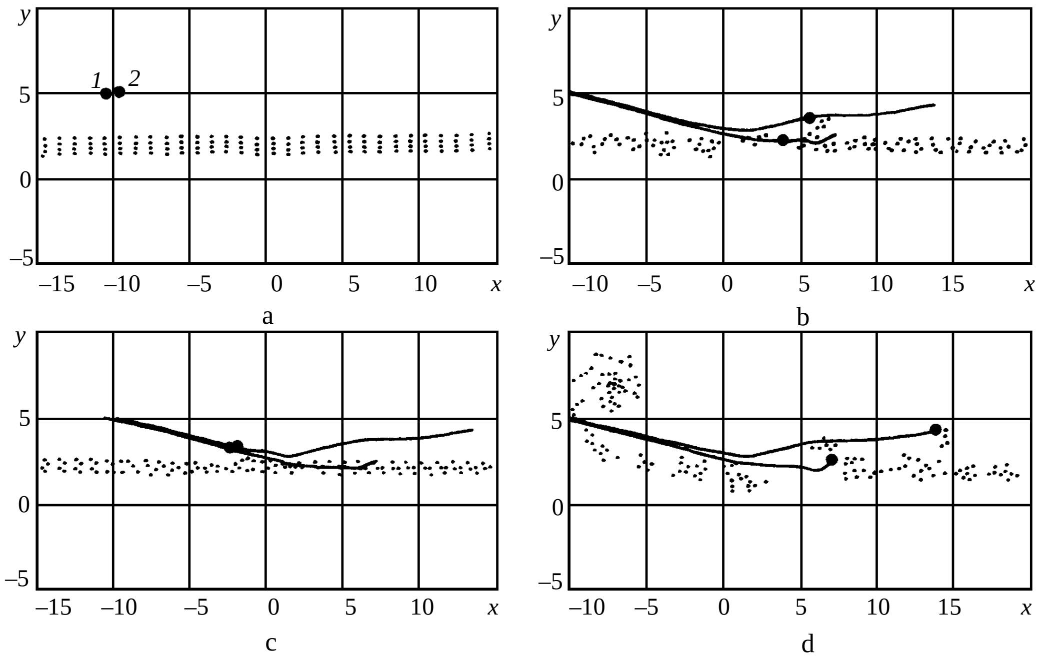

Fig. 8

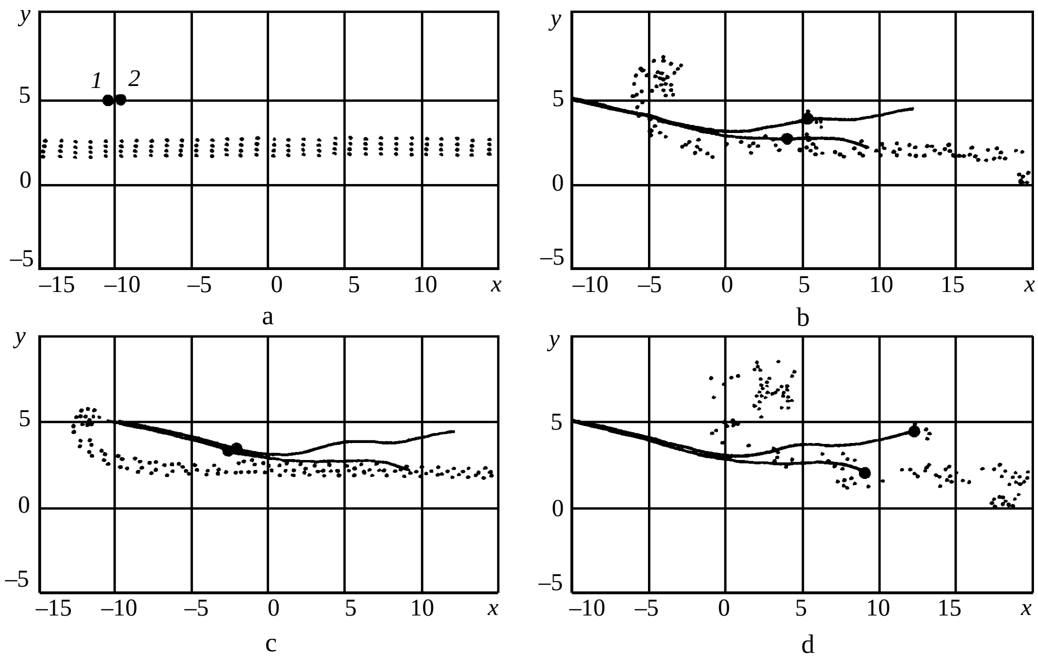

Fig. 9

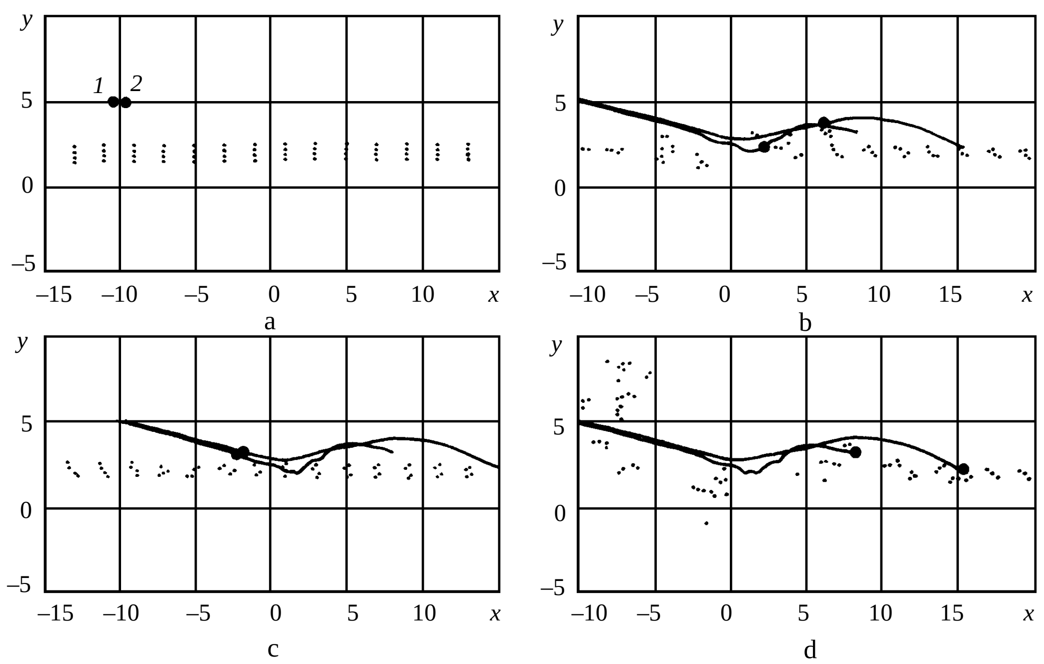

Fig. 10

that of the initial problem — the width of the active (moving) shear layer l and the number of point vortices $N \times M$ filling this region.

The model contains a distributed vorticity region located rather far from the initial vortex pair compared with the distance between the vortices. This region does not deform significantly all the time the interaction proceeds. Therefore, the model employs the following simplifying assumption: the layer with distributed vorticity does not move. However, this flow region should not be completely ignored. The absence of a shear layer, even far away, leads to instability of the shear layer — its middle part swirls eventually in accordance with the vorticity sign. As a result, the interaction of the shear layer with the vortex pair follows a different kind of scenario.

A comparison of the numerical results with the model approximation of pair–layer interaction and the numerical solution of a similar viscous problem shows that to describe the interaction process more accurately, it is necessary to form a greater number of layers of point vortices in the middle part of the shear flow. However, the excessive number of layers leads to a significant increase in the computing time. The optimal value for quite simple velocity profiles (when $U_x^{(0)}(z)$ is a linear or parabolic dependence) is $N \approx 3$–6.

The width l of the "cut out" part renders no significant effect on the evolution of the vortex system over a wide range of values. An analysis shows that the motionless part with distributed vorticity should be far from the interacting vortex pair.

In modeling the interaction, one should not decrease the number M of point vortices in each of the layers constituting the central part of the shear layer. The interaction of the vortex pair with the shear layer is modeled best when the distances between the vortices in the layer and between the layers in the middle part of the shear flow are of the same order.

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